

A DENSE G-DELTA SET OF RIEMANNIAN METRICS WITHOUT THE FINITE BLOCKING PROPERTY

MARLIES GERBER AND WAH-KWAN KU

ABSTRACT. A pair of points (x, y) in a Riemannian manifold (M, g) is said to have the finite blocking property if there is a finite set $P \subset M \setminus \{x, y\}$ such that every geodesic segment from x to y passes through a point of P . We show that for every closed C^∞ manifold M of dimension at least two and every pair $(x, y) \in M \times M$, there exists a dense G_δ set, \mathcal{G} , of C^∞ Riemannian metrics on M such that (x, y) fails to have the finite blocking property for every $g \in \mathcal{G}$.

1. INTRODUCTION

Let M be a closed C^∞ manifold, and let g be a C^∞ Riemannian metric on M . We consider a geodesic as a mapping $\gamma : I \rightarrow M$, where I is an interval of positive length, and γ is parametrized by arc length. Two geodesics $\gamma_i : I_i \rightarrow M$, $i = 1, 2$ will be considered to be the same if and only if $\gamma_1 = \gamma_2 \circ \varphi$, where φ is a translation that maps I_1 onto I_2 . Let x and y be points in M , possibly with $x = y$. When we say that a geodesic $\gamma : [c, d] \rightarrow M$ is from x to y , we mean $\gamma(c) = x$ and $\gamma(d) = y$.

Given a Riemannian metric g on M , a *blocking set* for (x, y) is defined to be a subset P of $M \setminus \{x, y\}$ such that every geodesic from x to y passes through a point in P . The pair $(x, y) \in M \times M$ is said to have the *finite blocking property* for g if there exists a finite blocking set for (x, y) . If every $(x, y) \in M \times M$ has the finite blocking property, then (M, g) is called *secure*. (See [7] and [4] for an explanation of this terminology.) A Riemannian manifold (M, g) is called *insecure* if it is not secure, and it is called *totally insecure* if no pair (x, y) has the finite blocking property. Furthermore, it is called *uniformly secure* if there exists a positive integer n such that any pair of points (x, y) has a blocking set with at most n elements.

Given a manifold M , it is natural to ask the following:

Question. Which pairs of points $(x, y) \in M \times M$ and which Riemannian metrics g on M are such that (x, y) has the finite blocking property for g ?

Our contribution in this direction is Theorem 1.1 below, which says that any given pair of points (x, y) fails to have the finite blocking property for a dense G_δ set of metrics. We will give the proof in Section 3.

We let \mathbb{G} denote the set of C^∞ Riemannian metrics on M . For $k = 1, 2, \dots, \infty$, there exists a complete metric on \mathbb{G} whose topology coincides with the C^k topology on \mathbb{G} . In particular, the Baire category theorem applies to \mathbb{G} with the C^k topology. When we refer to the C^k topology on $M \times \mathbb{G}$ or $M \times M \times \mathbb{G}$, we mean the product topology, where we take the manifold topology on M and the C^k topology on \mathbb{G} .

Theorem 1.1. *Let M be a closed C^∞ manifold of dimension at least two, and let \mathbb{G} be the space of C^∞ Riemannian metrics on M . The following three statements hold.*

- (1) Let x and y be two points in M , possibly with $x = y$. Let $\mathcal{G} = \{g \in \mathbb{G} : (x, y) \text{ fails to have the finite blocking property for } g\}$. Then \mathcal{G} contains the intersection of a countable collection of sets that are C^1 -open and C^∞ -dense in \mathbb{G} . Thus, \mathcal{G} contains a dense G_δ set in the C^k topology for $k = 1, 2, \dots, \infty$.
- (2) Let $\tilde{\mathcal{G}} = \{(x, y, g) \in M \times M \times \mathbb{G} : (x, y) \text{ fails to have the finite blocking property for } g\}$. Then $\tilde{\mathcal{G}}$ contains the intersection of a countable collection of sets that are C^1 -open and C^∞ -dense in $M \times M \times \mathbb{G}$. Thus, $\tilde{\mathcal{G}}$ contains a dense G_δ set in the C^k topology for $k = 1, 2, \dots, \infty$.
- (3) Let $\hat{\mathcal{G}} = \{(x, g) \in M \times \mathbb{G} : (x, x) \text{ fails to have the finite blocking property for } g\}$. Then $\hat{\mathcal{G}}$ contains the intersection of a countable collection of sets that are C^1 -open and C^∞ -dense in $M \times \mathbb{G}$. Thus, $\hat{\mathcal{G}}$ contains a dense G_δ set in the C^k topology for $k = 1, 2, \dots, \infty$.

From (2) and (3), we can deduce the following corollary.

Corollary 1.2. *Let M be a closed C^∞ manifold of dimension at least two and suppose $k \in \{1, 2, \dots, \infty\}$.*

- (1) *There exists a dense G_δ set \mathcal{G}_1 in \mathbb{G} with the C^k topology, so that for each $g \in \mathcal{G}_1$, there is a dense G_δ subset $\mathcal{R}_1 := \mathcal{R}_1(g)$ of $M \times M$ such that each $(x, y) \in \mathcal{R}_1$ fails to have the finite blocking property for g .*
- (2) *There exists a dense G_δ set \mathcal{G}_2 in \mathbb{G} with the C^k topology, so that for each $g \in \mathcal{G}_2$, there is a dense G_δ subset $\mathcal{R}_2 := \mathcal{R}_2(g) \subseteq M$ such that for each $x \in \mathcal{R}_2$, (x, x) fails to have the finite blocking property for g .*

V. Bangert and E. Gutkin obtained stronger results for the case when the dimension of M is two and the genus is positive [1]. They proved that if M has genus greater than one, then every Riemannian metric is totally insecure. Moreover, if M has genus one, they showed that non-flat metrics are insecure and a C^2 -open, C^∞ -dense set of metrics are totally insecure. These results provide evidence that (c) follows from (a) in the following conjecture, which originally appeared in [4] and [9]. A proof that (c) implies (b) is given in [8].

Conjecture 1.3. *Let (M, g) be a closed C^∞ Riemannian manifold. The following statements are equivalent.*

- (a) *(M, g) is secure.*
- (b) *(M, g) is uniformly secure.*
- (c) *g is a flat metric.*

While Conjecture 1.3 concerns the finite blocking property for *all pairs of points*, Theorem 1.1 shows that the finite blocking property can be destroyed for *any given pair of points*, under some small perturbation of metric.

In the next section, we will present some results which will be used to prove Theorem 1.1. We refer the reader to [6] for background information about geodesics and conjugate points.

We thank Chris Connell for a helpful conversation that led to an improvement to our original version of Theorem 1.1.

2. SOME PRELIMINARY RESULTS

We begin with the following classical result by J. P. Serre [12], [10], [2], [11].

Theorem 2.1. *Let (M, g) be a closed C^∞ -Riemannian manifold, and let $x, y \in M$. Then there exist infinitely many geodesics from x to y .*

The following lemma allows us to “merge” two foliations by geodesics for a Riemannian metric g into a new foliation by geodesics for a small perturbation of g , provided the two original foliations are C^∞ -close.

For $a, b > 0$, we let I_a denote the open interval $(-a, a) \subset \mathbb{R}$, and we let B_b denote the open ball $\{\mathbf{w} \in \mathbb{R}^{n-1} : |\mathbf{w}| < b\}$, where n is the dimension of the manifold M under consideration.

Lemma 2.2. *Let (M, g) be a closed C^∞ Riemannian manifold of dimension $n \geq 2$, and let \mathbb{G} be the set of C^∞ Riemannian metrics on M . Suppose \mathcal{N} is an open neighborhood of g in \mathbb{G} with the C^∞ topology. Choose $a, b > 0$, and let $\mathcal{F} = \{f \in C^\infty(I_a \times B_b, M) \mid f \text{ satisfies (i), (ii), (iii) below}\}$.*

- (i) *The map f is a C^∞ -diffeomorphism onto its image.*
- (ii) *For all $\mathbf{p} \in B_b$, the map $t \mapsto f(t, \mathbf{p})$, for $t \in I_a$, is a geodesic (for the metric g).*
- (iii) *For all $t \in I_a$, the $(n-1)$ -dimensional submanifold $\{f(t, \mathbf{p}) : \mathbf{p} \in B_b\}$ is perpendicular (in the metric g) to all the geodesics in (ii).*

We consider \mathcal{F} with the relative topology induced from the C^∞ compact-open topology on $C^\infty(I_a \times B_b, M)$. Suppose $f_0 \in \mathcal{F}$. Then there exists an open neighborhood $\mathcal{F}_0 \subseteq \mathcal{F}$ of f_0 such that for all $f_1, f_2 \in \mathcal{F}_0$, there exists $\tilde{g} \in \mathcal{N}$ such that the following conditions are satisfied.

- (1) \tilde{g} agrees with g on the complement of $f_1(I_{a/2} \times B_{b/2}) \cap f_2(I_{a/2} \times B_{b/2})$.
- (2) There is a family of \tilde{g} -geodesics $\gamma_{\mathbf{p}} : I_a \rightarrow f_1(I_a \times B_b) \cup f_2(I_a \times B_b)$, for $\mathbf{p} \in B_{b/4}$, such that

$$\gamma_{\mathbf{p}}(t) = \begin{cases} f_1(t, \mathbf{p}), & \text{if } t \in (-a, -a/4); \\ f_2(t, \mathbf{p}), & \text{if } t \in (a/4, a). \end{cases}$$

- (3) If $f_1(t, \mathbf{0}) = f_2(t, \mathbf{0})$ for all $t \in I_a$, then $\gamma_{\mathbf{0}}(t) = f_1(t, \mathbf{0})$. This implies that the map $t \mapsto f_1(t, \mathbf{0})$ for $t \in I_a$, is a geodesic for \tilde{g} as well.

Proof. Let $(a_i)_{0 \leq i \leq 5}$ and $(b_j)_{0 \leq j \leq 5}$ be strictly decreasing sequences of positive numbers, where $a_0 = a, a_3 = a/2, a_5 = a/4, b_0 = b, b_1 = b/2$, and $b_5 = b/4$. Let $R_{i,j} = I_{a_i} \times B_{b_j}$, for $0 \leq i, j \leq 5$.

Let $h : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ function such that

$$h(t) = \begin{cases} 0, & \text{if } t \leq -a_5; \\ 1, & \text{if } t \geq a_5, \end{cases}$$

and let $H : M \rightarrow [0, 1]$ be a C^∞ function such that

$$H(x) = \begin{cases} 0, & \text{if } x \in M \setminus f_0(R_{3,3}); \\ 1, & \text{if } x \in f_0(\overline{R_{4,4}}). \end{cases}$$

Given $f_0 \in \mathcal{F}$, the required open neighborhood \mathcal{F}_0 will be chosen so that functions $f_1, f_2 \in \mathcal{F}_0$ satisfy the properties given below. We begin by requiring f_1, f_2 to be sufficiently close to f_0 in the C^0 topology so that

$$(2.1) \quad f_2(\overline{R_{i+1,j+1}}) \subseteq f_1(R_{i,j}) \text{ and } f_1(\overline{R_{i+1,j+1}}) \subseteq f_2(R_{i,j}), \text{ for } 0 \leq i, j \leq 4.$$

We define $\phi : \overline{R_{1,1}} \rightarrow R_{0,0}$ by

$$\phi(t, \mathbf{p}) = (1 - h(t))(t, \mathbf{p}) + h(t)(f_1^{-1} \circ f_2(t, \mathbf{p}))$$

for $(t, \mathbf{p}) \in \overline{R_{1,1}}$, where ‘+’ denotes the usual vector addition in \mathbb{R}^n . We have $\phi(t, \mathbf{p}) \in R_{0,0}$, because $f_1^{-1} \circ f_2(\overline{R_{1,1}}) \subseteq R_{0,0}$ (by (2.1)), and $\phi(t, \mathbf{p})$ is a convex combination of $f_1^{-1} \circ f_2(t, \mathbf{p})$ and (t, \mathbf{p}) .

Next we consider $\hat{f} := f_1 \circ \phi : \overline{R_{1,1}} \rightarrow f_1(R_{0,0})$. If f_1 and f_2 are close to f_0 in $C^\infty(R_{0,0}, M)$, then ϕ is close to the inclusion map $\overline{R_{1,1}} \hookrightarrow R_{0,0}$ in $C^\infty(\overline{R_{1,1}}, R_{0,0})$, and \hat{f} is close to f_0 in $C^\infty(\overline{R_{1,1}}, M)$. We require f_1 and f_2 to be sufficiently close to f_0 in $C^\infty(R_{0,0}, M)$ so that the following four conditions are satisfied:

$$(2.2) \quad \hat{f} : \overline{R_{1,1}} \rightarrow M \text{ is a diffeomorphism onto its image,}$$

$$(2.3) \quad \hat{f}(\overline{R_{i+1,j+1}}) \subseteq f_0(R_{i,j}) \cap f_1(R_{i,j}) \cap f_2(R_{i,j}), \text{ for } 0 \leq i, j \leq 4,$$

$$(2.4) \quad f_0(\overline{R_{3,3}}) \subseteq \hat{f}(R_{2,2}), \text{ and}$$

$$(2.5) \quad (f_1((-a_1, -a_2] \times B_{b_2}) \cup f_2([a_2, a_1] \times B_{b_2})) \cap \hat{f}(\overline{R_{5,2}}) = \emptyset.$$

For $(t, \mathbf{p}) = (t, p_1, \dots, p_{n-1}) \in R_{2,2}$, we define a Riemannian metric \hat{g} at $\hat{f}(t, \mathbf{p}) \in \hat{f}(R_{2,2})$ by

$$(2.6) \quad \hat{g} \left(\frac{\partial \hat{f}}{\partial t}, \frac{\partial \hat{f}}{\partial t} \right) = 1,$$

$$(2.7) \quad \hat{g} \left(\frac{\partial \hat{f}}{\partial t}, \frac{\partial \hat{f}}{\partial y_k} \right) = 0, \text{ and}$$

$$\hat{g} \left(\frac{\partial \hat{f}}{\partial p_k}, \frac{\partial \hat{f}}{\partial p_l} \right) = [1 - h(t)]g \left(\frac{\partial f_1}{\partial p_k}, \frac{\partial f_1}{\partial p_l} \right) + h(t)g \left(\frac{\partial f_2}{\partial p_k}, \frac{\partial f_2}{\partial p_l} \right),$$

for $1 \leq k, l \leq n-1$.

We know that, for $i = 0, 1, 2$, the original metric g satisfies

$$g \left(\frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial t} \right) = 1, \text{ and}$$

$$g \left(\frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial p_k} \right) = 0, \text{ for } k = 1, \dots, n-1,$$

in the region $f_i(R_{0,0})$.

We define the required Riemannian metric as

$$\tilde{g} = H\hat{g} + (1 - H)g,$$

where we interpret $H\hat{g}$ to be 0 when $H = 0$.

If $(t, \mathbf{p}) \in [-a_1, -a_5] \times B_{b_1}$, then $h(t) = 0$ and $\phi(t, \mathbf{p}) = (t, \mathbf{p})$; if $(t, \mathbf{p}) \in [a_5, a_1] \times B_{b_1}$, then $h(t) = 1$ and $\phi(t, \mathbf{p}) = f_1^{-1} \circ f_2(t, \mathbf{p})$. Thus

$$(2.8) \quad \hat{f}(t, \mathbf{p}) = \begin{cases} f_1(t, \mathbf{p}), & \text{if } (t, \mathbf{p}) \in [-a_1, -a_5] \times B_{b_1}; \\ f_2(t, \mathbf{p}), & \text{if } (t, \mathbf{p}) \in [a_5, a_1] \times B_{b_1}. \end{cases}$$

Therefore \hat{g} agrees with g on $\hat{f}(R_{2,2} \setminus R_{5,2})$. If f_1 and f_2 are close to f_0 in $C^\infty(R_{0,0}, M)$, then \hat{g} is C^∞ -close to g on $\hat{f}(R_{2,2}) \supseteq f_0(\overline{R_{3,3}})$. Since $\tilde{g} = g$ on $M \setminus f_0(\overline{R_{3,3}})$, we may choose \mathcal{F}_0 sufficiently small so that $\tilde{g} \in \mathcal{N}$ for $f_1, f_2 \in \mathcal{F}_0$.

To summarize, we have chosen \mathcal{F}_0 sufficiently small so that if $f_1, f_2 \in \mathcal{F}_0$, then (2.1), (2.2), (2.3), (2.4), and (2.5) hold, and $\tilde{g} \in \mathcal{N}$.

Now we verify that (1), (2), and (3) hold.

The region where \hat{g} is defined and not equal to g is contained in $\hat{f}(R_{5,2})$, which is a subset of $f_1(R_{3,1}) \cap f_2(R_{3,1})$, by (2.3). Therefore $\tilde{g} = g$ on the complement of $f_1(R_{3,1}) \cap f_2(R_{3,1})$, which is conclusion (1).

Since $H = 1$ on $f_0(\overline{R_{4,4}}) \supseteq \hat{f}(R_{5,5})$, we have $\tilde{g} = \hat{g}$ on $\hat{f}(R_{5,5})$. For each $\mathbf{p} \in B_{b_5}$, we define a curve $\gamma_{\mathbf{p}} : I_a \rightarrow M$ as

$$\gamma_{\mathbf{p}}(t) = \begin{cases} f_1(t, \mathbf{p}), & \text{if } t \in (-a, -a_2]; \\ \hat{f}(t, \mathbf{p}), & \text{if } t \in (-a_2, a_2); \\ f_2(t, \mathbf{p}), & \text{if } t \in [a_2, a). \end{cases}$$

It follows from (2.8) that these curves are smooth. Moreover, these curves are \tilde{g} -geodesics, because $\tilde{g} = g$ on $f_1((-a, -a_2] \times B_{b_5}) \cup f_2([a_2, a) \times B_{b_5})$ (by (2.5)), $\hat{g} = g = \tilde{g}$ on $\hat{f}((\overline{I_{a_2}} \setminus I_{a_5}) \times B_{b_5}) = f_1([-a_2, -a_5] \times B_{b_5}) \cup f_2([a_5, a_2] \times B_{b_5})$, $\tilde{g} = \hat{g}$ on $\hat{f}(R_{5,5})$, and the curves $t \mapsto \hat{f}(t, \mathbf{p})$ are \hat{g} -geodesics for all $\mathbf{p} \in B_{b_2}$ (by (2.6) and (2.7)). This proves conclusion (2). If $f_1(t, \mathbf{0}) = f_2(t, \mathbf{0})$ for $t \in I_a$, then $\phi(t, \mathbf{0}) = (t, \mathbf{0})$ and $\hat{f}(t, \mathbf{0}) = f_1(t, \mathbf{0})$ for $t \in \overline{I_{a_1}}$. Therefore the \tilde{g} -geodesic $\gamma_{\mathbf{0}}$ is the same as $t \mapsto f_1(t, \mathbf{0})$, which establishes (3). \square

We now define a notion of *merging* for two geodesics. This will be used in Lemma 2.4 below.

Definition 2.3. Let M be a C^∞ -manifold, and let g, \tilde{g} be Riemannian metrics on M . Suppose U is an open set in M , $t_0 \in \mathbb{R}$, and $\gamma_i : [\hat{r}_i, \hat{s}_i] \rightarrow M$, $i = 1, 2$, are g -geodesics such that

$$(2.9) \quad \{t \in [\hat{r}_i, \hat{s}_i] : \gamma_i(t) \in U\} = (r_i, s_i), \text{ where } \hat{r}_i < r_i < t_0 < s_i < \hat{s}_i.$$

We say that a \tilde{g} -geodesic $\gamma : [\hat{r}_1, \hat{s}_2] \rightarrow M$, *merges* γ_1 and γ_2 *within* U if there exist \tilde{r}, \tilde{s} such that $r_1 < \tilde{r} < t_0 < \tilde{s} < s_2$, $\gamma(\tilde{r}, \tilde{s}) \subseteq U$, $\gamma(t) = \gamma_1(t)$ for $\hat{r}_1 \leq t \leq \tilde{r}$, and $\gamma(t) = \gamma_2(t)$ for $\tilde{s} \leq t \leq \hat{s}_2$.

The following lemma allows us to merge two geodesics according to Definition 2.3. K. Burns and G. Paternain have a similar result in the 2-dimensional case [5].

Lemma 2.4. Let (M, g) be a closed C^∞ Riemannian manifold of dimension $n \geq 2$, and let \mathcal{N} be an open neighborhood of g in the C^∞ topology. Suppose U is a convex (with respect to g) open set in M and $(x_0, v_0) \in T^1U$. Then there exists an open neighborhood \mathcal{V} of (x_0, v_0) in T^1U such that for any $(x_i, v_i) \in \mathcal{V}$, $i = 1, 2$, if $\gamma_i : [\hat{r}_i, \hat{s}_i] \rightarrow M$ are g -geodesics that satisfy (2.9) and $(\gamma_i(t_0), \gamma'_i(t_0)) = (x_i, v_i)$, for $i = 1, 2$, then there exists $\tilde{g} \in \mathcal{N}$ which agrees with g on $M \setminus U$, and a \tilde{g} -geodesic γ that merges γ_1 and γ_2 within U .

Proof. Let $\gamma_0 : [\hat{r}_0, \hat{s}_0] \rightarrow M$ be a g -geodesic such that $(\gamma_0(t_0), \gamma'_0(t_0)) = (x_0, v_0)$ and (2.9) is satisfied for $i = 0$ and some choice of r_0, s_0 . By replacing U by a smaller convex open neighborhood of x_0 , if necessary, we may assume there exist

C^∞ orthonormal vector fields E_1, \dots, E_n on U such that $E_n(\gamma_0(t)) = \gamma_0'(t)$ for all $t \in (r_0, s_0)$. We may assume that $t_0 = 0$. Choose T such that $0 < T < |r_0|$ and $\tilde{x}_0 := \gamma_0(-T)$ is not conjugate to x_0 along $\gamma_0|[-T, 0]$. For $u \in U$ and $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$, let

$$(2.10) \quad \Phi(u, \mathbf{z}) = z_1 E_1(u) + \dots + z_n E_n(u) \in T_u U.$$

Define $\varphi : \{\mathbf{p} = (p_1, \dots, p_{n-1}) \in \mathbb{R}^{n-1} : |\mathbf{p}| < 1\} \rightarrow \{\mathbf{w} \in \mathbb{R}^n : |\mathbf{w}| = 1\}$ by

$$(2.11) \quad \varphi(\mathbf{p}) = (p_1, \dots, p_{n-1}, 1 - (p_1^2 + \dots + p_{n-1}^2)^{1/2}).$$

Since \tilde{x}_0 and x_0 are not conjugate along $\gamma_0|[-T, 0]$, there exist $\tilde{a}, \tilde{b} > 0$ such that the map

$$f_0(t, \mathbf{p}) := \exp_{\tilde{x}_0}(\Phi(\tilde{x}_0, (t+T)\varphi(\mathbf{p}))),$$

defined for $(t, \mathbf{p}) \in I_{\tilde{a}} \times B_{\tilde{b}}$, is a C^∞ diffeomorphism onto its image, and its image is contained in U . Note that $f_0(0, \mathbf{0}) = x_0$. Moreover, there exist a, b with $0 < a < \tilde{a}$, $0 < b < \tilde{b}$, an open neighborhood \mathcal{A} of Id in $SO(n)$, and an open neighborhood \tilde{U} of \tilde{x}_0 in U such that for $\tilde{x} \in \tilde{U}$ and $A \in \mathcal{A}$, the map

$$f(t, \mathbf{p}) := \exp_{\tilde{x}}(\Phi(\tilde{x}, (t+T)A(\varphi(\mathbf{p})))),$$

defined for $(t, \mathbf{p}) \in I_a \times B_b$ is a C^∞ diffeomorphism onto its image, and its image is in U . Now choose \mathcal{V} to be an open neighborhood of (x_0, v_0) in $T^1 U$ such that for each $(x, v) \in \mathcal{V}$, the geodesic $\tilde{\gamma}$ with $(\tilde{\gamma}(0), \tilde{\gamma}'(0)) = (x, v)$ satisfies $\tilde{x} := \tilde{\gamma}(-T) \in \tilde{U}$ and there exists $A \in \mathcal{A}$ with $\Phi(\tilde{x}, A(\varphi(\mathbf{0}))) = \tilde{\gamma}'(-T)$. We also require \mathcal{V} to be small enough so that \tilde{x} is sufficiently close to \tilde{x}_0 and A can be chosen sufficiently close to Id , so that f is in the neighborhood \mathcal{F}_0 of f_0 given in Lemma 2.2. (The hypothesis (iii) in Lemma 2.2 for f_0 , as well as f_1, f_2 defined below, follows from the Gauss Lemma.)

Let $(x_i, v_i) \in \mathcal{V}$, $i = 1, 2$, and suppose $\gamma_i : [\hat{r}_i, \hat{s}_i] \rightarrow M$, $i = 1, 2$, are g -geodesics such that (2.9) is satisfied and $(\gamma_i(0), \gamma_i'(0)) = (x_i, v_i)$. Let r_i, s_i , $i = 1, 2$, be as in (2.9). For $i = 1, 2$, define

$$f_i(t, \mathbf{p}) := \exp_{\tilde{x}_i}(\Phi(\tilde{x}_i, (t+T)A_i(\varphi(\mathbf{p})))),$$

for $(t, \mathbf{p}) \in I_a \times B_b$, where $\tilde{x}_i := \gamma_i(-T)$, and $A_i \in \mathcal{A}$ is such that $\Phi(\tilde{x}_i, A_i(\varphi(\mathbf{0}))) = \gamma_i'(-T)$. Then $f_i(t, \mathbf{0}) = \gamma_i(t)$ for $t \in I_a$. From Lemma 2.2, we obtain $\tilde{g} \in \mathcal{N}$ which agrees with g on $M \setminus U$ so that conclusion (2) of Lemma 2.2 holds. Finally, we define the required \tilde{g} -geodesic $\gamma : [\hat{r}_1, \hat{s}_2] \rightarrow M$ as

$$\gamma(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [\hat{r}_1, -a]; \\ \gamma_0(t), & \text{if } t \in (-a, a); \\ \gamma_2(t), & \text{if } t \in [a, \hat{s}_2], \end{cases}$$

where γ_0 is as in Lemma 2.2(2). \square

Lemma 2.5 below allows us to destroy conjugate points along a geodesic by making a small perturbation of the metric. A two-dimensional version of this lemma is contained in [5].

Lemma 2.5. *Let (M, g) be a closed C^∞ Riemannian manifold of dimension $n \geq 2$, and let \mathcal{N} be an open neighborhood of g in the C^∞ topology. Let $x, y \in M$ and suppose $\gamma : [0, L] \rightarrow M$ is a g -geodesic from x to y . Let $0 = t_0 < t_1 < \dots < t_\ell = L$, where $\ell \geq 1$, and define $z_k := \gamma(t_k)$ for $k = 0, \dots, \ell$. Suppose $s_0 \in (t_j, t_{j+1})$ for*

some $j \in \{0, \dots, \ell - 1\}$ and $u_0 := \gamma(s_0)$ is not a self-intersection point of γ (i.e., $u_0 \notin \gamma([0, T] \setminus \{s_0\})$). Let U_0 be an open neighborhood of u_0 . Then there exists $\hat{g} \in \mathcal{N}$ that agrees with g on $M \setminus U_0$ such that the following conditions hold:

- (1) γ is also a unit speed geodesic for \hat{g} .
- (2) If k_1 and k_2 are integers such that $0 \leq k_1 \leq j$ and $j + 1 \leq k_2 \leq \ell$, then z_{k_1} is not conjugate to z_{k_2} along $\gamma|_{[t_{k_1}, t_{k_2}]}$ in the \hat{g} metric.

Proof. It suffices to prove the lemma for the case $\ell = 1$ and $0 = t_0 < s_0 < t_1 = L$, because we can then obtain (2) in the general case through a finite sequence of perturbations of the metric (within \mathcal{N}) corresponding to each possible pair (k_1, k_2) with $0 \leq k_1 \leq j$ and $j + 1 \leq k_2 \leq \ell$. Each successive perturbation adds one more pair (k_1, k_2) such that z_{k_1} is not conjugate to z_{k_2} along $\gamma|_{[t_{k_1}, t_{k_2}]}$, and the perturbations can be taken small so that no new conjugacies are introduced between such pairs of points.

We now assume $\ell = 1$ and $0 = t_0 < s_0 < t_1 = L$. By perturbing s_0 slightly, if necessary, we may assume that x is not conjugate to u_0 along $\gamma|_{[0, s_0]}$. We may also assume that the open neighborhood U_0 of u_0 is chosen so that $\{t \in [0, L] : \gamma(t) \in U_0\} = (s_0 - \eta, s_0 + \eta)$ for some η with $0 < \eta < \min(s_0, t_1 - s_0)$. Let U be an open neighborhood of x disjoint from U_0 . Suppose $\tau \in (0, s_0 - \eta)$ is such that $\gamma|_{[0, \tau]}$ is one-to-one, and whenever $0 < t \leq \tau$, x is not conjugate to $\gamma(t)$ along $\gamma|_{[0, t]}$, and $\gamma(t)$ is not conjugate to y along $\gamma|_{[t, L]}$. Let E_1, \dots, E_n be C^∞ vector fields along $\gamma|_{[0, \tau]}$ with $\gamma'(t) = E_n(\gamma(t))$ for $t \in [0, \tau]$. Let Φ and φ be as in (2.10) and (2.11) for $u \in \gamma([0, \tau])$. Since x is not conjugate to u_0 along $\gamma|_{[0, s_0]}$, there exist $\tilde{a}, \tilde{b} > 0$ such that the map

$$f_1(t, \mathbf{p}) := \exp_{x,g}(\Phi(x, (t + s_0)\varphi(\mathbf{p})),$$

defined for $(t, \mathbf{p}) \in I_{\tilde{a}} \times B_{\tilde{b}}$, is a C^∞ diffeomorphism onto its image, and its image is in U_0 . (The ‘ g ’ in the subscript indicates we are referring to the exponential map for the metric g .) There exist $a, b, \tilde{\delta}$ with $0 < a < \tilde{a}$, $0 < b < \tilde{b}$, $0 < \tilde{\delta} < \tau$, such that the map

$$f_2(t, \mathbf{p}) := \exp_{\tilde{x},g}(\Phi(\tilde{x}, (t + s_0 - \delta)\varphi(\mathbf{p})),$$

defined for $(t, \mathbf{p}) \in I_a \times B_b$ is a C^∞ diffeomorphism onto its image, and its image is in U_0 for any $\tilde{x} := \gamma(\delta)$ with $0 < \delta < \tilde{\delta}$. Let f_0 be the restriction of f_1 to $I_a \times B_b$, and let \mathcal{F}_0 be as in Lemma 2.2. We choose δ sufficiently small so that $f_2 \in \mathcal{F}_0$. Since $f_1(I_{a/2} \times B_{b/2}) \cap f_2(I_{a/2} \times B_{b/2})$ is a subset of U_0 , Lemma 2.2 implies that there is a $\hat{g} \in \mathcal{N}$ which agrees with g on $M \setminus U_0$ and Lemma 2.2(2) holds with \tilde{g} replaced by \hat{g} . We also obtain Lemma 2.2(3) with \tilde{g} replaced by \hat{g} , because $f_1(t, \mathbf{0}) = f_2(t, \mathbf{0})$ for $t \in I_a$. Therefore γ is also a geodesic for \hat{g} . For $\mathbf{p} \in B_{b/4}$, let $\gamma_{\mathbf{p}}$ be as in Lemma 2.2(2) and define $\sigma_{\mathbf{p}} : [0, L] \rightarrow M$ by

$$(2.12) \quad \sigma_{\mathbf{p}}(t) = \begin{cases} \exp_{x,g}(\Phi(x, t\varphi(\mathbf{p}))), & \text{if } t \in [0, s_0 - a]; \\ \gamma_{\mathbf{p}}(t - s_0), & \text{if } t \in (s_0 - a, s_0 + a); \\ \exp_{\tilde{x},g}(\Phi(\tilde{x}, (t - \delta)\varphi(\mathbf{p}))), & \text{if } t \in [s_0 + a, L]. \end{cases}$$

Then $\sigma_{\mathbf{p}}$ is a \hat{g} -geodesic that merges, within U_0 , a \hat{g} -geodesic originating at x with initial velocity $\Phi(x, \varphi(\mathbf{p}))$ and a g -geodesic that is at \tilde{x} with velocity $\Phi(\tilde{x}, \varphi(\mathbf{p}))$ at time δ . Thus, for $\mathbf{p} \in B_{b/4}$,

$$(2.13) \quad \exp_{x,\hat{g}}(\Phi(x, t\varphi(\mathbf{p}))) = \exp_{\tilde{x},g}(\Phi(\tilde{x}, (t - \delta)\varphi(\mathbf{p})))$$

for $s_0 + a \leq t \leq L$. Since \tilde{x} is not conjugate to y along $\gamma|[\delta, L]$ in the metric g , $\exp_{\tilde{x},g}$ is locally a diffeomorphism near $(L - \delta)\gamma'(\delta)$. By (2.13), this implies that $\exp_{x,\hat{g}}$ is locally a diffeomorphism near $L\gamma'(0)$. Therefore x is not conjugate to y along γ in the \hat{g} metric. \square

A *geodesic lasso* is defined to be a closed curve which is a geodesic except at one point, where it fails to be regular. The following Lemma 2.6 allows us to perturb a geodesic so that it avoids a finite set of points on M , and it also allows us to change a closed geodesic to a geodesic lasso.

Lemma 2.6. *Let (M, g) be a closed C^∞ Riemannian manifold of dimension $n \geq 2$, and let \mathcal{N} be an open neighborhood of g in the C^∞ topology. Let $x, y \in M$ and suppose $\gamma : [0, L] \rightarrow M$ is a g -geodesic from x to y . Let Z be a finite set of points in M such that $x, y \in Z$. Let $\{t \in [0, L] : \gamma(t) \in Z\} = \{t_k : k = 0, \dots, \ell\}$, where $0 = t_0 < \dots < t_\ell = L$, $\ell \geq 1$, and define $z_k := \gamma(t_k)$, for $k = 0, \dots, \ell$. Assume that*

- (i) *x is not conjugate to z_k along $\gamma| [0, t_k]$, for $k = 1, \dots, \ell$.*
- (ii) *z_k is not conjugate to y along $\gamma| [t_k, L]$, for $k = 0, \dots, \ell - 1$.*

Suppose $s_0 \in (0, L)$, $u_0 := \gamma(s_0)$ is not a self-intersection point of γ , and $u_0 \notin Z$. Let U_0 be an open neighborhood of u_0 . Then there exist open neighborhoods W_1 and W_2 of $\gamma'(0)$ and $\gamma'(L)$ in $T_x^1 M$ and $T_y^1 M$, respectively, such that for any $w_1 \in W_1 \setminus \{\gamma'(0)\}$ and any $w_2 \in W_2 \setminus \{\gamma'(L)\}$, there exists $\tilde{g} \in \mathcal{N}$ that agrees with g on $M \setminus U_0$ and a \tilde{g} -geodesic $\tilde{\gamma} : [0, L] \rightarrow M$ from x to y such that $\tilde{\gamma}'(0) = w_1$, $\tilde{\gamma}'(L) = w_2$, $\tilde{\gamma}([0, L]) \cap Z = \emptyset$, and x is not conjugate to y along $\tilde{\gamma}$ for \tilde{g} .

Proof. We may assume that $Z \subset \gamma([0, L])$. By replacing U_0 by a smaller open neighborhood of u_0 if necessary, we may assume that U_0 is convex for g , $U_0 \cap Z = \emptyset$, and $\{t \in [0, L] : \gamma(t) \in U_0\} = (s_0 - \eta, s_0 + \eta)$, for some $\eta > 0$.

Since x is not conjugate to z_k along $\gamma| [0, t_k]$ for $k = 1, \dots, \ell$, and $\exp_{x,g}$ is locally a diffeomorphism near $0 \in T_x M$, there exist neighborhoods V_k of $t_k \gamma'(0)$ in $T_x M$, for $k = 0, \dots, \ell$, such that the maps $\exp_{x,g} : V_k \rightarrow M$ are diffeomorphisms onto their images. Also,

$$(2.14) \quad Z \cap \exp_{x,g}(\{t\gamma'(0) : t \in [0, L]\} \setminus (V_0 \cup \dots \cup V_\ell)) = \emptyset,$$

because $(\exp_{x,g}^{-1} Z) \cap \{t\gamma'(0) : t \in [0, L]\} = \{t_0\gamma'(0), \dots, t_\ell\gamma'(0)\}$. By the continuity of $\exp_{x,g}$, we can choose W_1 sufficiently small so that (2.14) still holds for γ replaced by any g -geodesic $\gamma_1 : [0, L] \rightarrow M$ with $\gamma_1(0) = x$ and $\gamma_1'(0) \in W_1$. If $\gamma_1'(0) \in W_1 \setminus \{\gamma'(0)\}$, then $\{t\gamma_1'(0) : t \in (0, L]\} \cap (V_0 \cup \dots \cup V_\ell)$ does not contain any of $t_k\gamma'(0)$, $k = 0, \dots, \ell$. Thus, (2.14) for γ_1 implies that $\gamma_1([0, L]) \cap Z = \emptyset$. Similarly, if W_2 is sufficiently small, then for any g -geodesic $\gamma_2 : [0, L] \rightarrow M$ with $\gamma_2(L) = y$ and $\gamma_2'(L) \in W_2 \setminus \{\gamma'(L)\}$, we have $\gamma_2([0, L]) \cap Z = \emptyset$.

Let $v_0 = \gamma'(s_0)$ and let \mathcal{V} be an open neighborhood of (u_0, v_0) in $T^1 U_0$ satisfying the conclusion of Lemma 2.4 (with U replaced by U_0 and x_0 replaced by u_0). In addition to the requirements of the preceding paragraph, we require W_1 and W_2 to be sufficiently small so that if $\gamma_i : [0, L] \rightarrow M$, $i = 1, 2$, are such that $\gamma_1(0) = x$, $\gamma_1'(0) \in W_1$, $\gamma_2(L) = y$, and $\gamma_2'(L) \in W_2$, then there exist r_i, s_i with $0 < r_i < s_0 < s_i < L$, such that $\{t \in [0, L] : \gamma_i(t) \in U_0\} = (r_i, s_i)$ and $(\gamma_i(s_0), \gamma_i'(s_0)) \in \mathcal{V}$.

Suppose $w_1 \in W_1 \setminus \{\gamma'(0)\}$ and $w_2 \in W_2 \setminus \{\gamma'(L)\}$, and let $\gamma_i : [0, L] \rightarrow M$, $i = 1, 2$, be g -geodesics such that $\gamma_1(0) = x$, $\gamma_1'(0) = w_1$, $\gamma_2(L) = y$, and $\gamma_2'(L) = w_2$. By Lemma 2.4, there exists a metric $\tilde{g} \in \mathcal{N}$ that agrees with g on $M \setminus U_0$ and a \tilde{g} -geodesic $\tilde{\gamma} : [0, L] \rightarrow M$ that merges γ_1 and γ_2 within U_0 . Since $U_0 \cap Z = \emptyset$ and

$\gamma_1((0, L)) \cap Z = \emptyset = \gamma_2((0, L)) \cap Z$, we have $\tilde{\gamma}((0, L)) \cap Z = \emptyset$. By Lemma 2.5 we can make a small additional perturbation of the metric \tilde{g} within U_0 , if necessary, to arrange for x and y to not be conjugate along $\tilde{\gamma}$. \square

3. PROOF OF THEOREM 1.1

We now use the results of Section 2 to prove Theorem 1.1. The notation $\text{tr}(\gamma)$ will mean the trace of a curve $\gamma : I \rightarrow M$, i.e., $\text{tr}(\gamma) = \{\gamma(t) : t \in I\}$.

Proof. Let $(x, y, g) \in M \times M \times \mathbb{G}$, and let $n \in \mathbb{N}$. We consider the statement $S(x, y, n, g)$: there exist g -geodesics $\gamma_i : [0, L_i] \rightarrow M$ from x to y , $i = 1, \dots, n$, which satisfy the following four properties:

- (i) If $x \neq y$, then the set of tangent vectors

$$\{\gamma'_1(0), \gamma'_2(0), \dots, \gamma'_n(0)\}$$

at x are pairwise linearly independent, and the set of tangent vectors

$$\{\gamma'_1(L_1), \gamma'_2(L_2), \dots, \gamma'_n(L_n)\}$$

at y are pairwise linearly independent. If $x = y$, then the set of tangent vectors

$$\{\gamma'_1(0), \gamma'_1(L_1), \gamma'_2(0), \gamma'_2(L_2), \dots, \gamma'_n(0), \gamma'_n(L_n)\}$$

are pairwise linearly independent. Thus we cannot join γ_i to γ_j smoothly at x or at y , for any $i, j \in \{1, \dots, n\}$.

- (ii) For each $i = 1, \dots, n$, we have $\gamma_i((0, L_i)) \cap \{x, y\} = \emptyset$. That is, γ_i meets x and y only at its endpoints.
- (iii) Any three of $\gamma_1, \dots, \gamma_n$ are concurrent only at x and at y .
- (iv) The point x is not conjugate to y in the metric g along $\gamma_i|_{[0, L_i]}$, for $i = 1, \dots, n$.

We define $\mathcal{H}_n(x, y) := \{g \in \mathbb{G} : S(x, y, n, g) \text{ is satisfied}\}$. We make the following claim:

Claim 3.1. (a) $\mathcal{H}_n(x, y)$ is C^∞ -dense in \mathbb{G} and (b) there is a C^1 -open neighborhood $\mathcal{G}_n(x, y)$ of $\mathcal{H}_n(x, y)$ in \mathbb{G} such that parts (i), (ii), and (iii) of $S(x, y, n, g)$ are satisfied for all $g \in \mathcal{G}_n(x, y)$.

Claim 3.1 implies that the set $\bigcap \mathcal{G}_n(x, y)$ is a dense G_δ set for \mathbb{G} with the C^k topology, for $k = 1, 2, \dots, \infty$. Suppose $P \subseteq M \setminus \{x, y\}$ is a set with m points, and $g \in \bigcap \mathcal{G}_n(x, y)$. Since $g \in \mathcal{G}_{2m+1}$, we can find $2m+1$ g -geodesics that satisfy (iii). If P were a blocking set for (x, y) , then by the pigeonhole principle, at least three of these geodesics would pass through the same point in P , which leads to contradiction. Hence there is no finite blocking set for (x, y) , and Theorem 1.1(1) follows from Claim 3.1.

Similarly, if we define $\tilde{\mathcal{H}}_n := \{(x, y, g) \in M \times M \times \mathbb{G} : S(x, y, n, g) \text{ is satisfied}\}$ and $\tilde{\mathcal{H}}_n := \{(x, g) \in M \times \mathbb{G} : S(x, x, n, g) \text{ is satisfied}\}$, and we prove the following claims, then Theorem 1.1(2),(3) will follow by considering $\bigcap \tilde{\mathcal{G}}_n$ and $\bigcap \hat{\mathcal{G}}_n$, respectively.

Claim 3.2. (a) $\tilde{\mathcal{H}}_n$ is C^∞ -dense in $M \times M \times \mathbb{G}$ and (b) there is a C^1 -open neighborhood $\tilde{\mathcal{G}}_n$ of $\tilde{\mathcal{H}}_n$ in $M \times M \times \mathbb{G}$ such that (i), (ii), and (iii) of $S(x, y, n, g)$ are satisfied for all $(x, y, g) \in \tilde{\mathcal{H}}_n$.

Claim 3.3. (a) $\widehat{\mathcal{H}}_n$ is C^∞ -dense in $M \times \mathbb{G}$ and (b) there is a C^1 -open neighborhood $\widehat{\mathcal{G}}_n$ of $\widehat{\mathcal{H}}_n$ in $M \times \mathbb{G}$ such that (i), (ii), and (iii) of $S(x, x, n, g)$ are satisfied for all $(x, g) \in \widehat{\mathcal{H}}_n$.

We now prove Claim 3.1(a) by mathematical induction. For $n = 1$, let \mathcal{N} be any non-empty C^∞ -open set in \mathbb{G} , and let $g \in \mathcal{N}$. Let $\gamma : [0, L] \rightarrow M$ be a g -geodesic from x to y . By restricting the domain of γ , if necessary, we may assume that $\gamma((0, L)) \cap \{x, y\} = \emptyset$. Then we let $\ell = 1$ and $0 = t_0 < t_1 = L$ in Lemma 2.5. By Lemma 2.5, there exists $\hat{g} \in \mathcal{N}$ such that γ is also a unit speed geodesic for \hat{g} and x is not conjugate to y along γ . If $(x, \gamma'(0)) \neq (y, \gamma'(L))$, then we let $g_1 = \hat{g}$, and $\gamma_1 = \gamma$. If $(x, \gamma'(0)) = (y, \gamma'(L))$, that is, γ is a closed geodesic, then we apply Lemma 2.6 to obtain $g_1 \in \mathcal{N}$ and a g_1 -geodesic lasso $\gamma_1 : [0, L] \rightarrow M$ with $\gamma_1(0) = \gamma_1(L) = x$ but $\gamma_1'(0) \neq \gamma_1'(L)$, and $x \notin \gamma_1((0, L))$. Then (i), (ii), and (iv) are satisfied, and (iii) is vacuous. Since \mathcal{N} is arbitrary, $\mathcal{H}_1(x, y)$ is C^∞ -dense.

Next we suppose $\mathcal{H}_{n-1}(x, y)$ is C^∞ -dense for some $n \geq 2$, and we will prove that $\mathcal{H}_n(x, y)$ is C^∞ -dense. Let \mathcal{N} be any non-empty C^∞ -open set in \mathbb{G} . There exist $g_{n-1} \in \mathcal{H}_{n-1}(x, y) \cap \mathcal{N}$ and g_{n-1} -geodesics $\gamma_i : [0, L_i] \rightarrow M$ from x to y , $i = 1, \dots, n-1$, so that properties (i) - (iv) are satisfied with n replaced by $n-1$. By Theorem 2.1, there exists a g_{n-1} -geodesic $\gamma : [0, L] \rightarrow M$ from x to y , distinct from $\gamma_1, \dots, \gamma_{n-1}$. If $x = y$, we also require γ to be distinct from $-\gamma_1, \dots, -\gamma_{n-1}$, where $-\gamma_i$ is γ_i traversed in the opposite direction.

By (i) and (ii), we have $\text{tr}(\gamma) \not\subseteq \text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})$. However, it may happen that $\text{tr}(\gamma)$ contains one (or more) of the sets $\text{tr}(\gamma_1), \dots, \text{tr}(\gamma_{n-1})$. If $x = y$, then we can restrict the domain of γ , if necessary, so that $\text{tr}(\gamma)$ does not contain any of the sets $\text{tr}(\gamma_1), \dots, \text{tr}(\gamma_{n-1})$. If $x \neq y$, then we can restrict the domain of γ , if necessary, to obtain a g_{n-1} -geodesic from x to y such that one of the following happens: (a) $\text{tr}(\gamma)$ does not contain any of the sets $\text{tr}(\gamma_1), \dots, \text{tr}(\gamma_{n-1})$; (b) γ consists of one of $\gamma_1, \dots, \gamma_{n-1}$ preceded by a g_{n-1} -geodesic from x to x ; (c) γ consists of one of $\gamma_1, \dots, \gamma_{n-1}$ followed by a g_{n-1} -geodesic from y to y . If (a) holds, then we assume that $\text{tr}(\gamma)$ does not contain any of the sets $\text{tr}(\gamma_1), \dots, \text{tr}(\gamma_{n-1})$, and the rest of this paragraph can be skipped. So assume that one of cases (b) or (c) hold, and assume that the domain of γ has been restricted so that cases (b) and (c) do not hold for any further restriction to a proper closed subinterval of the domain. Let $u_0 \in \text{tr}(\gamma) \setminus [\text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})]$ be such that u_0 is not a self-intersection point of γ , and let U_0 be an open neighborhood of u_0 such that $U_0 \cap [\text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})] = \emptyset$. By Lemma 2.5, we can make a perturbation of the g_{n-1} metric within U_0 such that γ remains a geodesic, the new metric is in \mathcal{N} , and neither of x or y is conjugate to either of x or y along an arc of γ . Then Lemma 2.6 applies with $Z = \{x, y\}$. Thus we may again perturb the metric within U_0 to produce a new metric $\hat{g} \in \mathcal{N}$ and a \hat{g} -geodesics $\hat{\gamma}$ close to γ and different from $\gamma_1, \dots, \gamma_{n-1}$, such that $\hat{\gamma}$ meets x and y only at its endpoints. In particular, $\text{tr}(\gamma)$ does not contain any of the sets $\text{tr}(\gamma_1), \dots, \text{tr}(\gamma_{n-1})$. Since $U_0 \cap [\text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})] = \emptyset$, $\gamma_1, \dots, \gamma_{n-1}$ remain geodesics for \hat{g} .

From the preceding paragraph, we have a metric $\hat{g} \in \mathcal{N}$ and a \hat{g} -geodesic $\hat{\gamma} : [0, L] \rightarrow M$ from x to y such that $\gamma_1, \dots, \gamma_{n-1}$ are \hat{g} -geodesics and $\text{tr}(\hat{\gamma})$ does not contain any of the sets $\text{tr}(\gamma_1), \dots, \text{tr}(\gamma_{n-1})$. Then $\text{tr}(\hat{\gamma}) \cap [\text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})]$ is a finite set. If $n = 2$, let $Z = \{x, y\}$; if $n > 2$, let Z be the collection of all intersection points between the trace of any two of $\gamma_1, \dots, \gamma_{n-1}$. From (i) and (ii), we know that Z is a finite set. We also have $x, y \in Z$. We want to perturb $\hat{\gamma}$ so

that it does not meet Z except at its endpoints. Let $\hat{\gamma}^{-1}(Z) \cap [0, L] = \{t_0, \dots, t_\ell\}$, where $0 = t_0 < \dots < t_\ell = L$, and denote $z_k := \hat{\gamma}(t_k)$, for $k = 0, \dots, \ell$. Let $s_1 \in (t_0, t_1)$, $s_2 \in (t_{\ell-1}, t_\ell)$, $s_1 < s_2$, $u_1 := \hat{\gamma}(s_1)$, $u_2 := \hat{\gamma}(s_2)$ be such that $u_1, u_2 \notin \text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})$ and u_1, u_2 are not self-intersection points of $\hat{\gamma}$. We can apply Lemma 2.5 twice with $s_0 = s_i$ and $U_0 = U_i$ for $i = 1, 2$, where $(U_1 \cup U_2) \cap [\text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})] = \emptyset$. Thus we obtain a metric $\bar{g} \in \mathcal{N}$ such that $\gamma_1, \dots, \gamma_{n-1}$ are \bar{g} -geodesics, and conditions (i) and (ii) in Lemma 2.6 hold for g replaced by \bar{g} and γ replaced by $\hat{\gamma}$. Hence, by Lemma 2.6, there is a metric $\tilde{g} \in \mathcal{N}$ such that $\gamma_1, \dots, \gamma_{n-1}$ are \tilde{g} -geodesics, and there is a \tilde{g} -geodesic $\tilde{\gamma}$ from x to y that is different from $\gamma_1, \dots, \gamma_{n-1}$, and does not meet any point of Z except at its endpoints. Moreover, by Lemma 2.6, we may choose \tilde{g} and $\tilde{\gamma}$ so that x and y are not conjugate along $\tilde{\gamma}$ in the \tilde{g} -metric. All of the perturbations of the metric can be done outside a neighborhood of $\text{tr}(\gamma_1) \cup \dots \cup \text{tr}(\gamma_{n-1})$. We let $g_n = \tilde{g}$. Then $\gamma_1, \dots, \gamma_{n-1}$ are g_n -geodesics, and (iv) remains true for $\gamma_1, \dots, \gamma_{n-1}$ with the metric g_n . Thus properties (i)-(iv) hold for $\gamma_1, \dots, \gamma_n$, where $\gamma_n = \tilde{\gamma}$, and g is replaced by g_n . Since \mathcal{N} is arbitrary, we conclude that $\mathcal{H}_n(x, y)$ is C^∞ -dense. This completes the proof of Claim 3.1(a).

Claim 3.2(a) and Claim 3.3(a) follow from Claim 3.1(a), because $\tilde{\mathcal{H}}_n$ is C^∞ -dense in each fiber $\{(x, y)\} \times \mathbb{G}$, and $\hat{\mathcal{H}}_n$ is C^∞ -dense in each fiber $\{x\} \times \mathbb{G}$.

Next we want to prove Claim 3.1(b). Let $g \in \mathcal{H}_n(x, y)$, and suppose $\gamma_1, \dots, \gamma_n$ are g -geodesics that satisfy properties (i)-(iv).

If we consider geodesics as curves in T^1M , then they are solutions to a system of first order ordinary differential equations whose coefficients depend only on the first derivatives of the metric. For the purpose of defining C^1 distances between the given geodesics γ_i and nearby curves $\tilde{\gamma}_i$, we extend the domain of γ_i to $[0, L_i + 1]$. The C^1 distance will be measured with respect to the natural metric on T^1M induced by g . For any $\epsilon > 0$ there exists a C^1 neighborhood \mathcal{N}_1 of g in \mathbb{G} and a $\delta = \delta(\epsilon) > 0$ such that: if $\hat{g} \in \mathcal{N}_1$ and $\tilde{\gamma}_i$ is a \hat{g} -geodesic with $\tilde{\gamma}_i(0) = \gamma_i(0)$ and $|\tilde{\gamma}_i'(0) - \gamma_i'(0)| < \delta$, then the C^1 distance between $\tilde{\gamma}_i|[0, L_i + 1]$ and $\gamma_i|[0, L_i + 1]$ is less than ϵ . We choose $\epsilon > 0$ such that if the C^1 distance between $\tilde{\gamma}_i|[0, L_i + 1]$ and $\gamma_i|[0, L_i + 1]$ is less than ϵ , $|L_i - \tilde{L}_i| < \epsilon$, and $\tilde{\gamma}_i(\tilde{L}_i) = y$, then conditions (i)-(iii) hold with γ_i replaced by $\tilde{\gamma}_i$, and L_i replaced by \tilde{L}_i .

By (iv), y is not g -conjugate to x along any of $\gamma_1, \dots, \gamma_n$. We choose open neighborhoods U_1, \dots, U_n of $L_1\gamma_1'(0), \dots, L_n\gamma_n'(0)$ in T_xM , respectively, and an open neighborhood U of y in M , so that

$$\exp_{x,g} : U_i \rightarrow U$$

is a local diffeomorphism, for $i = 1, \dots, n$. By replacing U and U_i by smaller open neighborhoods, if necessary, we may assume that if $\tilde{\gamma}_i'(0)\tilde{L}_i \in U_i$, then $|L_i - \tilde{L}_i| < \epsilon$ and $|\tilde{\gamma}_i'(0) - \gamma_i'(0)| < \delta$.

If $B_i \subset U_i$ is an open ball centered at $L_i\gamma_i'(0)$ with $\overline{B_i} \subset U_i$, then $y \notin \exp_{x,g}(\partial B_i)$ and the topological degree of $\exp_{x,g}|_{\partial B_i}$ is nonzero at y . Any continuous map $f_i : \overline{B_i} \rightarrow U$ that is sufficiently close to $\exp_{x,g}|_{\overline{B_i}}$ in the C^0 topology also satisfies $y \notin f_i(\partial B_i)$, and the topological degree of $f_i|_{\partial B_i}$ is nonzero at y . This implies $y \in f_i(B_i)$. (See, for instance, Theorem 1.1 of [3].) Now we choose a C^1 -open neighborhood $\mathcal{N}_2 \subset \mathcal{N}_1$ of g such that if $\tilde{g} \in \mathcal{N}_2$, then $\exp_{x,\tilde{g}}$ is sufficiently C^0 -close to $\exp_{x,g}$ on $\overline{B_i}$, $i = 1, \dots, n$, so that there exist $y_i \in B_i$ with $\exp_{x,\tilde{g}} y_i = y$. For $\tilde{g} \in \mathcal{N}_2$, let $\tilde{\gamma}_i, i = 1, \dots, n$, be \tilde{g} -geodesics defined on $[0, \tilde{L}_i]$ such that $\tilde{\gamma}_i'(0)\tilde{L}_i = y_i$.

Then conditions (i)-(iii) hold for γ_i, L_i, g replaced by $\tilde{\gamma}_i, \tilde{L}_i, \tilde{g}$, respectively. Thus there exists a C^1 -open neighborhood \mathcal{G}_n of \mathcal{H}_n such that conditions (i)-(iii) hold for all $\tilde{g} \in \mathcal{G}_n$.

This finishes the proof of Claim 3.1(b), and thus the proof of Theorem 1.1(1). The proofs of Claims 3.2(b) and 3.3(b) are similar to the proof of Claim 3.1(b), except we do not assume that $\tilde{\gamma}_i(0) = \gamma_i(0)$. This completes the proof of Theorem 1.1.

□

REFERENCES

- [1] V. Bangert and E. Gutkin, *Insecurity for compact surfaces of positive genus*, Preprint, arXiv:0908.1138.
- [2] M. Berger, *A panoramic view of Riemannian geometry*, Springer-Verlag, Berlin, 2003.
- [3] F. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), no. 1, 1-39.
- [4] K. Burns and E. Gutkin, *Growth of the number of geodesics between points and insecurity for Riemannian manifolds*, Discrete Contin. Dyn. Syst. **21** (2008), no. 2, 403–413.
- [5] K. Burns and G. Paternain, *On the growth of the number of geodesics joining two points*, International Conference on Dynamical Systems, Montevideo (1995), Pitman Res. Notes Math. Ser. **362**, 1996, 7–20.
- [6] M. do Carmo, *Riemannian geometry*, Birkhäuser, Boston, 1992.
- [7] E. Gutkin, *Blocking of billiard orbits and security for polygons and flat surfaces*, Geom. Funct. Anal. **15** (2005), no. 1, 83–105.
- [8] E. Gutkin and V. Schroeder, *Connecting geodesics and security of configurations in compact locally symmetric spaces*, Geom. Dedicata **118** (2006), 185–208.
- [9] J.-F. Lafont and B. Schmidt, *Blocking light in compact Riemannian manifolds*, Geom. Topol. **11** (2007), 867–887.
- [10] J. Milnor, *Morse theory*, Princeton University Press, Princeton, 1969.
- [11] G. Paternain, *Geodesic flows*, Birkhäuser, Boston, 1999.
- [12] J.-P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math. (2) **54**, (1951), 425–505.

E-mail address: gerber@indiana.edu, wku@indiana.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405, USA